JOURNAL OF APPROXIMATION THEORY 50, 175–184 (1987)

Comparison of Bernstein Polynomials with the Metrical Means of Kantorovitch

ERICH VAN WICKEREN

Lehrstuhl A für Mathematik, RWTH Aachen, Templergraben 55, D-5100 Aachen, West Germany

Communicated by P. L. Butzer

Received June 13, 1985; revised July 18, 1985

DEDICATED TO THE MEMORY OF GÉZA FREUD

In 1934 Kantorovitch modified the Bernstein polynomials B_n by means of metrical means to yield a nonlinear polynomial process B_n^* which approximates measurable functions almost everywhere. The present paper is concerned with the pointwise comparison of B_n and B_n^* on $C[0, 1]$ (the space of continuous functions on $[0, 1]$). We establish direct estimates of the form $|(B_n f - f)(x)| \leqslant |(B_n^* f - f)(x)| + \omega(3n^{-1/3}, f)$ with the first modulus of continuity ω. On the other hand, it is the main purpose of this paper to show that this inequality can not be strengthened to $|(B_n f - f)(x)| \leqslant C_x |(B_n f - f)(x)|$ so that B_n^* is not a pointwise extension of B_n on $C[0, 1]$. To this end, a previous condensation principle is applied concerning nonlinear functionals which assures the nonvalidity of this last inequality for x on a dense, in fact, residual, set of $[0, 1]$. © 1987 Academic Press, Inc.

1. INTRODUCTION

For $f \in C[0, 1]$, the space of continuous functions on $[0, 1]$ (endowed with the usual sup-norm $\|\cdot\|_C$), it is well known that the Bernstein polynomials ($n \in \mathbb{N}$:= set of natural numbers)

$$B_n f := \sum_{k=0}^{n} f(k/n)\, p_{kn}, \qquad p_{kn}(x) := \binom{n}{k} x^k (1-x)^{n-k},$$

converge uniformly to f, i.e.,

$$\lim_{n \to \infty} \|B_n f - f\|_C = 0.$$

Kantorovitch [11] modified these polynomials in the form that the point

0021-9045/87 $3.00

functionals $f(k/n)$ are substituted by metrical means given via ($\mathbb{R} :=$ set of reals, $E \subset [0, 1]$ with Lebesgue measure meas $E > 0$)

$$\begin{aligned} M(E, f) &:= \sup\{u \in \mathbb{R} : m(u; E, f) \geqslant \tfrac{1}{2} \operatorname{meas} E\}, \\ m(u; E, f) &:= \operatorname{meas}\{v \in E : f(v) \geqslant u\}. \end{aligned} \tag{1.1}$$

Indeed, this yields pointwise convergence a.e. also for measurable functions f by the polynomials

$$\begin{aligned} B_n^* f &:= \sum_{k=0}^{n} M(E_{kn}, f)\, p_{kn}, \\ E_{kn} &:= \left[\frac{k}{n} - \delta_n, \frac{k}{n} + \delta_n\right] \cap [0, 1], \qquad \delta_n = 3n^{-1/3}. \end{aligned} \tag{1.2}$$

In this paper we consider the (pointwise) comparison of the two processes B_n and B_n^* on $C[0, 1]$, in particular the problem whether B_n^* is indeed an extension of B_n. Thereby it should be noted that B_n^* is not linear, even not sublinear, so that comparison theorems for (sub-) linear operators do not apply here.

In Section 2 we first establish direct estimates of the form

$$\begin{aligned} |(B_n^* f - f)(x)| &\leqslant |(B_n f - f)(x)| + \omega(\delta_n, f), \\ |(B_n f - f)(x)| &\leqslant |(B_n^* f - f)(x)| + \omega(\delta_n, f) \end{aligned} \tag{1.3}$$

where ω is the first modulus of continuity (cf. (2.9)). On the other hand, it is shown in Section 3 that (1.3) can *not* be strengthened to

$$|(B_n^* f - f)(x)| \leqslant C_{x,f}\, |(B_n f - f)(x)| \leqslant C_{x,f}^*\, |(B_n^* f - f)(x)| \tag{1.4}$$

for some constants $C_{x,f}$, $C_{x,f}^*$, independent of $n \in \mathbb{N}$. This will be achieved by a quantitative condensation principle (cf. Theorem 3.2) developed in [13; 14]; it delivers a counterexample $f_0 \in C[0, 1]$ such that both inequalities of (1.4) are false even for x of a dense, indeed residual set of $[0, 1]$. Moreover, this condensation principle simultaneously yields the sharpness of (cf. Theorem 2.2)

$$|(B_n^* f - f)(x)| \leqslant 3\omega(\delta_n, f)$$

for the same counterexample f_0 and for x on the same dense set.

Besides these results let us mention that the techniques used in Section 3 show how one may obtain condensation of singularities in a concrete (but nonlinear) situation by using resonance and condensation principles developed in a general frame work (see [3–8; 13; 14] and the literature cited there, also for further applications).

2. Direct Comparison Theorems

Let us first establish some elementary properties of the (nonsublinear) functional $M(E, f)$.

Lemma 2.1. *Let* $f \in C[0,1]$ *and* $E \subset [0,1]$ *with* meas $E > 0$.

(a) *$M(E, f)$ is uniquely determined by*

$$m(M(E, f); E, f) \geqslant \tfrac{1}{2} \operatorname{meas} E, \tag{2.1}$$

$$m(M(E, f) + \varepsilon; E, f) < \tfrac{1}{2} \operatorname{meas} E \qquad (\varepsilon > 0) \tag{2.2}$$

so that for $u \in \mathbb{R}$

$$m(u; E, f) < \tfrac{1}{2} \operatorname{meas} E \Rightarrow M(E, f) < u, \tag{2.3}$$

$$m(u; E, f) \geqslant \tfrac{1}{2} \operatorname{meas} E \Rightarrow M(E, f) \geqslant u. \tag{2.4}$$

(b) *For* $b \geqslant 0$, $c \in \mathbb{R}$ *one has*

$$M(E, c) = c, \tag{2.5}$$

$$M(E, f + c) = M(E, f) + c, \tag{2.6}$$

$$M(E, bf) = bM(E, f), \tag{2.7}$$

$$|f(v) - g(v)| \leqslant b \qquad \text{for all } v \in E \Rightarrow |M(E, f) - M(E, g)| \leqslant b. \tag{2.8}$$

Proof. Obviously, $u_0 := M(E, f)$ is well defined since f is bounded. Now $m(u; E, f)$ is decreasing and continuous in u (cf. [1, p. 23]) so that there exists $u_n > u_0 - 1/n$ such that

$$\tfrac{1}{2} \operatorname{meas} E \leqslant m(u_n; E, f) \leqslant m(u_0 - 1/n; E, f) \rightarrow m(u_0; E, f).$$

Hence the supremum in (1.1) is attained at u_0 so that one has (2.1), (2.2), and thus (2.3), (2.4), respectively, as well as the uniqueness. Moreover, (2.5) is obvious whereas (2.6) follows by substituting $x = u - c$ so that

$$M(E, f + c) = \sup\{x + c : m(x; E, f) \geqslant \tfrac{1}{2} \operatorname{meas} E\} = M(E, f) + c.$$

Analogously, one has (2.7) with $x = u/b$ $(b > 0)$. If $f(v) \leqslant g(v)$ for $v \in E$, then by (2.1)

$$\tfrac{1}{2} \operatorname{meas} E \leqslant m(M(E, f); E, f) \leqslant m(M(E, f); E, g)$$

so that $M(E, f) \leqslant M(E, g)$ by (2.4). Hence, (2.8) is a consequence of (2.6) and

$$f(v) - b \leqslant g(v) \leqslant f(v) + b \qquad (v \in E). \quad \blacksquare$$

Obviously, B_n^* is uniformly bounded on $C[0, 1]$ in view of (2.8) (take $g = 0$, $b = \|f\|_C$). But this implies the continuity of $B_n^* f$ only at $f = 0$ since B_n^* is non-subadditive. However, the following theorem establishes continuity on the whole space, in fact in terms of the first modulus of continuity

$$\omega(t, f) := \sup\{|f(x) - f(x+h)| : x, x+h \in [0, 1], |h| \leqslant t\}. \tag{2.9}$$

THEOREM 2.2. *For $n \in \mathbb{N}$ the operator $B_n^* f - f$ is continuous on $C[0, 1]$, in fact ($f, g \in C[0, 1]$, cf. (1.2))*

$$\|(B_n^* f - f) - (B_n^* g - g)\|_C \leqslant 3\omega(\delta_n, f - g). \tag{2.10}$$

In particular, one has the direct approximation estimate

$$\|B_n^* f - f\|_C \leqslant 3\omega(\delta_n, f). \tag{2.11}$$

Proof. Let us first note that (cf. [2, p. 26])

$$\sum_{k=0}^{n} p_{kn}(x) = 1, \tag{2.12}$$

$$\sum_{|k/n - x| > \delta_n} p_{kn}(x) \leqslant \delta_n/108. \tag{2.13}$$

For $x \in [0, 1]$ one has in view of (2.8) for $0 \leqslant k \leqslant n$, $v \in E_{kn}$,

$$|v - x| \leqslant |v - k/n| + |k/n - x| \leqslant \delta_n + |k/n - x|,$$

$$|(f(v) - f(x)) - (g(v) - g(x))| \leqslant \omega(\delta_n + |k/n - x|, f - g),$$

$$|M(E_{kn}, f - f(x)) - M(E_{kn}, g - g(x))| \leqslant \omega(\delta_n + |k/n - x|, f - g).$$

Therefore (2.10) follows by (2.6), (2.12), (2.13),

$$\begin{aligned}
&|(B_n^* f - f)(x) - (B_n^* g - g)(x)| \\
&\quad \leqslant \sum_{k=0}^{n} p_{kn}(x)\, |M(E_{kn}, f - f(x)) - M(E_{kn}, g - g(x))| \\
&\quad \leqslant \sum_{|k/n - x| \leqslant \delta_n} p_{kn}(x)\, \omega(2\delta_n, f - g) + \sum_{|k/n - x| > \delta_n} p_{kn}(x)\, \omega(1, f - g) \\
&\quad \leqslant \omega(2\delta_n, f - g) + (\delta_n/108)\, \omega(1, f - g) \leqslant 3\omega(\delta_n, f - g),
\end{aligned}$$

the latter inequality being a consequence of (cf. [12, p. 99])

$$\omega(s, f)/s \leqslant 2\omega(t, f)/t \qquad (0 < t < s).$$

Thus $B_n^* f - f$ is continuous, and (2.11) follows by setting $g = 0$ (cf. (2.5)). ∎

To compare B_n, B_n^* (in pointwise sense) one may proceed analogously to deduce

THEOREM 2.3. *Let* $n \in \mathbb{N}$, $x \in [0, 1]$, *and* $f \in C[0, 1]$. *Then* (1.3) *holds true.*

Proof. Since $|f(v) - f(k/n)| \leqslant \omega(\delta_n, f)$ for $0 \leqslant k \leqslant n$, $v \in E_{kn}$, one obtains in view of (2.6), (2.8),

$$|M(E_{kn}, f) - f(k/n)| \leqslant \omega(\delta_n, f)$$

so that (1.3) follows, noting that by (2.12)

$$|(B_n^* f - B_n f)(x)| \leqslant \sum_{k=0}^{n} |M(E_{kn}, f) - f(k/n)|\, p_{kn}(x) \leqslant \omega(\delta_n, f). \quad ∎$$

Of course, (1.3) does not deliver direct comparison estimates of B_n and B_n^* on account of the additional term $\omega(\delta_n, f)$. However, in the following section it will be shown that these inequalities cannot be strengthened to the form (1.4).

3. CONDENSATION OF SINGULARITIES WITH RATES

Let ω be a positive function on $(0, \infty)$ such that $(0 < s < t)$

$$\text{(i)}\quad \omega(s) \leqslant \omega(t), \qquad \text{(ii)}\quad \lim_{t \to 0+} \omega(t) = 0, \tag{3.1}$$

$$\text{(i)}\quad s/\omega(s) \leqslant t/\omega(t), \qquad \text{(ii)}\quad \lim_{t \to 0+} t^{1/3}/\omega(t) = 0 \tag{3.2}$$

(e.g., $\omega(t) = t^{\alpha}$, $0 < \alpha < \frac{1}{3}$). Moreover, let $A_\omega = \{A_n\}$, $A_\omega^* = \{A_n^*\}$ be positive sequences tending to infinity such that

$$\lim_{n \to \infty} A_n / n^{1/2} \omega(1/n) = 0, \tag{3.3}$$

$$\lim_{n \to \infty} A_n^* / n^{1/3} \omega(1/n) = 0 \tag{3.4}$$

(e.g., $A_n = (n^{1/2}\omega(1/n))^\beta$, $A_n^* = (n^{1/3}\omega(1/n))^\gamma$, $0 < \beta,\ \gamma < 1$). The following theorem establishes the sharpness of (2.11) (cf. (3.5), (3.6)) as well as the nonvalidity of (1.4) (cf. (3.7), (3.8)), pointwise on a residual set in [0, 1]. Recall that A is a residual set in [0, 1] if it is the complement of a set of first (Baire) category. Then A is dense and of second (Baire) category by Baire's theorem.

THEOREM 3.1. *For any* ω, A_ω, A_ω^* *subject to* (3.1)–(3.4), *respectively, there exists a function* $f_\omega \in C[0, 1]$ *and a residual set* $A_\omega \subset [0, 1]$ *such that for each* $x \in A_\omega$

$$\omega(t, f_\omega) \leqslant 4\omega(t) \qquad (t > 0), \tag{3.5}$$

$$\limsup_{n \to \infty} |(B_n^* f_\omega - f_\omega)(x)| / \omega(\delta_n) \geqslant 1, \tag{3.6}$$

$$\limsup_{n \to \infty} \frac{|(B_n^* f_\omega - f_\omega)(x)|}{|(B_n f_\omega - f_\omega)(x)|} \frac{1}{A_n} \geqslant 1, \tag{3.7}$$

$$\limsup_{n \to \infty} \frac{|(B_n f_\omega - f_\omega)(x)|}{|(B_n^* f_\omega - f_\omega)(x)|} \frac{1}{A_n^*} \geqslant 1. \tag{3.8}$$

The proof of this theorem is based on the following condensation principle with rates developed in [13; 14] by means of Baire category arguments (here it is quoted for the particular Banach space $C[0, 1]$): let I be a countable index set, let A_i, $i \in I$, be dense, countable subsets of [0, 1], and set

$$\text{Lip } 1 := \{ f \in C[0, 1] : \omega(t, f) = \mathcal{O}_f(t),\ t \to 0+ \}.$$

For $i \in I$ consider sequences $\{T_{ni}\}$, $\{V_{ni}\}$ of continuous operators on $C[0, 1]$ into itself (not necessarily sublinear).

THEOREM 3.2. *Let* $h_{ni} \in \text{Lip } 1$ *be given such that for each* $i \in I$, $x \in A_i$, $f \in \text{Lip } 1$ ($M > 0$, $0 < c_i \leqslant \infty$, $0 < b_i < \infty$, $0 < \varphi_{ni} = o_i(1)$ *for* $n \to \infty$)

$$\sup_{n \in \mathbb{N}} \|h_{ni}\|_C =: N_i < \infty, \tag{3.9}$$

$$\omega(t, h_{ni}) \leqslant M \min\{1, t/\varphi_{ni}\} \qquad (n \in \mathbb{N}, t > 0), \tag{3.10}$$

$$\limsup_{n \to \infty} |[T_{ni}(\omega(\varphi_{ni})\, h_{ni} + f)](x)| / \omega(\varphi_{ni}) \geqslant c_i, \tag{3.11}$$

$$\limsup_{n \to \infty} |[V_{ni}(\omega(\varphi_{ni})\, h_{ni} + f)](x)| / \omega(\varphi_{ni}) \leqslant b_i. \tag{3.12}$$

Then there exists $f_\omega \in C[0, 1]$ *and a residual set* $\Lambda_\omega \subset [0, 1]$ *such that for each* $x \in \Lambda_\omega$,

$$\omega(t, f_\omega) \leqslant M\omega(t) \qquad (t > 0), \tag{3.13}$$

$$\limsup_{n \to \infty} \frac{|T_{ni} f_\omega(x)|}{\max\{b_i \omega(\varphi_{ni}), |V_{ni} f_\omega(x)|\}} \geqslant \frac{c_i}{b_i}. \tag{3.14}$$

Proof of Theorem 3.1. Consider the dense countable index sets ($I = \{1, 2, 3\}$)

$$\Lambda_1 = \{6p/q \in (0, 1): p, q \in \mathbb{N}\}, \qquad \Lambda_2 = \{p/2^q \in (0, 1): p, q \in \mathbb{N}\},$$
$$\Lambda_3 = \{(p + 1/4)/5^q \in (0, 1): p, q \in \mathbb{N}\},$$

and the continuous operators (cf. Theorem 2.2)

$$T_{n1} f = B_n^* f - f, \qquad V_{n1} f = 0,$$
$$T_{n2} f = B_{2^n}^* f - f, \qquad V_{n2} f = A_{2^n}(B_{2^n} f - f),$$
$$T_{n3} f = B_{5^n} f - f, \qquad V_{n3} f = A_{5^n}^*(B_{5^n}^* f - f).$$

To construct the elements h_{ni} consider the 1-periodic function H given by $H(v) = 4\,|v|$, $|v| \leqslant \frac{1}{2}$, so that H is continuous with $0 \leqslant H(v) \leqslant 2$. Then one has for $0 \leqslant u \leqslant 1$, $a \in \mathbb{R}$ by substituting $z = v - a - \frac{1}{2}$

$$m(u; [a, a+1], H(v)) = m(u: [-\tfrac{1}{2}, \tfrac{1}{2}], H(z)) = 1 - u/2. \tag{3.15}$$

Moreover, for $a \in (q-1, q]$, $q \in \mathbb{N}$, it follows that

$$\begin{aligned} m(u; [-a, a], H) &= 2m(u; [0, a], H) \\ &= 2 \sum_{p=1}^{q-1} m(u; [p-1, p], H) + 2m(u; [q-1, a], H) \\ &\leqslant 2(q-1)(1-u/2) + 2 < 2a(1-u/2) + 2 \end{aligned}$$

or

$$\geqslant 2(q-1)(1-u/2) \geqslant 2a(1-u/2) - 2. \tag{3.16}$$

Next we will show that ($x \in (0, 1)$, $n \to \infty$),

$$[B_n^* H(nv)](x) = 1 + \mathcal{O}(n^{-1/3}), \tag{3.17}$$

$$[B_n^* H(v/2\delta_n)](x) = 1 + \mathcal{O}(n^{-1/3}). \tag{3.18}$$

To this end, let $n \in \mathbb{N}$ be such that $2\delta_n \leqslant x \leqslant 1-2\delta_n$. Consider first those k with $\delta_n \leqslant k/n \leqslant 1-\delta_n$. Then one obtains with $z = nv - k$ and (3.16)

$$m\left(1+\frac{2}{3}(-1)^r n^{-2/3}; E_{kn}, H(nv)\right)$$
$$=\frac{1}{n} m\left(1+\frac{2}{3}(-1)^r n^{-2/3}; [-3n^{2/3}, n^{2/3}], H(z)\right)\begin{cases} <\delta_n, & r=0, \\ \geqslant \delta_n, & r=1, \end{cases}$$

which implies by (2.3), (2.4),

$$1-\tfrac{2}{3}n^{-2/3} \leqslant M(E_{kn}, H(nv)) < 1+\tfrac{2}{3}n^{-2/3}.$$

For the other k one has $|k/n - x| \geqslant \delta_n$ so that in view of (2.12), (2.13),

$$[B_n^* H(nv)](x) = 1 + \mathcal{O}(n^{-2/3}) + \mathcal{O}\left(\sum_{|k/n - x| \geqslant \delta_n} p_{kn}(x)\right) = 1 + \mathcal{O}(n^{-1/3}).$$

Similarly one obtains (3.18) since with $z = v/2\delta_n$ and (3.15)

$$m(u; E_{kn}, H(v/2\delta_n)) = 2\delta_n m(u; [k/6n^{2/3} - \tfrac{1}{2}, k/6n^{2/3} + \tfrac{1}{2}], H(z))$$
$$= (1 - u/2) \text{ meas } E_{kn},$$
$$M(E_{kn}, H(v/2\delta_n)) = 1 \qquad (\delta_n \leqslant k/n \leqslant 1-\delta_n).$$

Now set $h_{ni}(v) = H(v/\varphi_{ni})$ with

$$\varphi_{n1} = 2\delta_n, \qquad \varphi_{n2} = 2^{-n}, \qquad \varphi_{n3} = 5^{-n}$$

so that (3.9) follows with $N_i = 2$ and (3.10) with $M = 4$. To verify (3.11), (3.12), $b_i = c_i = 1$ (the situation for V_{n1} is trivial) consider first the (linear) Bernstein polynomials. Let $x = p/2^q \in A_3$ and $n \geqslant q$. Then

$$h_{n2}(x) = H(p2^{n-q}) = 0, \tag{3.19}$$

in particular, $h_{n2}(k/2^n) = 0$ so that $(V_{n2} h_{n2})(x) = 0$. In view of (cf. [2, p. 27])

$$|(B_n f - f)(x)| \leqslant \tfrac{3}{2}\omega(n^{-1/2}, f) = \mathcal{O}(n^{-1/2}) \qquad (f \in \text{Lip } 1) \tag{3.20}$$

and (3.3) this yields (3.12) since

$$|[V_{n2}(\omega(\varphi_{n2})\, h_{n2} + f)(x)]| / \omega(\varphi_{n2}) \leqslant |(V_{n2} f)(x)| / \omega(\varphi_{n2}) = o(1).$$

On the other hand, for $x=(p+1/4)/5^q \in A_3$ one has $(n \geqslant q+1)$

$$x/\varphi_{n3} = p5^{n-q} + \sum_{j=0}^{n-q-1} 5^j + \frac{1}{4} =: K_n + \frac{1}{4},$$

$$h_{n3}(x) = H(K_n + \tfrac{1}{4}) = H(\tfrac{1}{4}) = 1 \tag{3.21}$$

so that $|(T_{n3}h_{n3})(x)| = 1$, since $h_{n3}(k/5^n) = H(k) = 0$. Thus (3.11) follows by (3.2)(ii), (3.20), since

$$|[T_{n3}(\omega(\varphi_{n3})\, h_{n3} + f)](x)|/\omega(\varphi_{n3})$$
$$\geqslant |(T_{n3}h_{n3})(x)| + \mathcal{O}(5^{-n/2}/\omega(5^{-n})) = 1 + o(1).$$

Concerning the metrical means, note first that in view of (2.7), (2.10), (3.2)(ii), (3.4)

$$|[T_{ni}(\omega(\varphi_{ni})\, h_{ni} + f)](x)|/\omega(\varphi_{ni})$$
$$\geqslant \begin{cases} |(T_{n1}h_{n1})(x)| + \mathcal{O}(\delta_n/\omega(2\delta_n)) = |(T_{n1}h_{n1})(x)| + o(1) \\ |(T_{n2}h_{n2})(x)| + \mathcal{O}(2^{-n/3}/\omega(2^{-n})) = |(T_{n2}h_{n2})(x)| + o(1), \end{cases}$$

$$|[V_{n3}(\omega(\varphi_{n3})\, h_{n3} + f)](x)|/\omega(\varphi_{n3})$$
$$\leqslant |(V_{n3}h_{n3})(x)| + \mathcal{O}(A^*_{5^n} 5^{-n/3}/\omega(5^{-n})) = |(V_{n3}h_{n3})(x)| + o(1).$$

Then for $x = 6p/q \in A_1$ and for the subsequence $n_j = 2^{3j}q^3$ it follows by (3.18) that

$$h_{n_j1}(x) = H(6pq2^j/pq) = 0, \qquad (T_{n_j1}h_{n_j1})(x) = 1 + o(1).$$

Moreover, by (3.1)(ii), (3.4), (3.17), (3.19), (3.21),

$$(T_{n2}h_{n2})(x) = 1 + o(1) \qquad (x \in A_2),$$
$$|(V_{n3}h_{n3})(x)| = \mathcal{O}(A^*_{2^n} 5^{-n/3}) = o(1) \qquad (x \in A_3)$$

so that the metrical means also fulfill (3.11), (3.12). Thus (3.13), (3.14) yield the assertions. ∎

Note that (3.7), (3.8) are valid simultaneously for the same x and f_ω because the limites superiores may be realized by different subsequences of $\mathbb{N}$, depending on x.

Finally, let us mention that uniform boundedness and condensation principles (but *without* rates) were developed in [9; 10] for nonlinear operators, too. This led to the definition of asymptotically subadditive operators in [9; 10] whereby the metrical means were one of the typical examples. However, an application was not given there since the operators

B_n^* are uniformly bounded on $C[0, 1]$. But, as it is shown here, the condensation principle Theorem 3.2 can be applied to this process for the Lipschitz class (3.5).

ACKNOWLEDGMENTS

The author would like to express his sincere gratitude to Werner Dickmeis and Rolf J. Nessel for their critical reading of the manuscript and many valuable suggestions. This work was supported by Deutsche Forschungsgemeinschaft Grant Ne 171/5.

REFERENCES

1. H. BAUER, "Wahrscheinlichkeitstheorie und Grundzüge der Maßtheorie," De Gruyter, Berlin, 1974.
2. P. L. BUTZER AND H. BERENS, "Semi-Groups of Operators and Approximation," Springer-Verlag, Berlin, 1967.
3. W. DICKMEIS, On quantitative condensation of singularities on sets of full measure, to appear.
4. W. DICKMEIS AND R. J. NESSEL, A unified approach to certain counterexamples in approximation theory in connection with a uniform boundedness principle with rates, *J. Approx. Theory* **31** (1981), 161–174.
5. W. DICKMEIS AND R. J. NESSEL, Condensation principles with rates, *Studia Math.* **75** (1982), 55–68.
6. W. DICKMEIS, R. J. NESSEL, AND E. VAN WICKEREN, A general approach to counterexamples in numerical analysis, *Numer. Math.* **43** (1984), 249–263.
7. W. DICKMEIS, R. J. NESSEL, AND E. VAN WICKEREN, On the sharpness of estimates in terms of averages, *Math. Nachr.* **117** (1984), 263–272.
8. W. DICKMEIS, R. J. NESSEL, AND E. VAN WICKEREN, On nonlinear condensation principles with rates, *Manuscripta Math.* **52** (1985), 1–20.
9. I. S. GÁL, Sur la methode de résonance et sur un théorème concernant les espaces de type (*B*), *Ann. Inst. Fourier (Grenoble)* **3** (1951), 23–50.
10. I. S. GÁL, The principle of condensation of singularities, *Duke Math. J.* **20** (1953), 27–35.
11. L. KANTOROVITCH, La représentation explicite d'une fonction mesurable arbitraire dans la forme de la limite d'une suite de polynómes, *Mat. Sb.* **41** (1934), 503–506.
12. A. F. TIMAN, "Theory of Approximation of Functions of a Real Variable," Pergamon, New York, 1963.
13. E. VAN WICKEREN, Resonanzprinzipien der Approximationstheorie aus Baire'scher Sicht, dissertation, RWTH, Aachen, 1984.
14. E. VAN WICKEREN, A Baire approach to quantitative resonance principles. *Numer. Funct. Anal. Optim.* (1986), in press.